Final for Queueing Theory (closed book) 6/21/1994

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1. (12%) We now have an M/G/1 system with $\lambda = 1/4$, and

$$B^*(s) = \frac{3}{(s+1)(4s+3)}$$

(a) (1%) Find ρ.
(b) (4%) Find $Q(Z)$.
(c) (2%) Find P_k=P[k in system in equilibrium].
(d) (3%) Find $W^*(s)$.
(e) (2%) Find $w(y)$.

2. (12%) Consider a "lazy" M/G/1 queueing system with arrival rate λ in which the server goes on vacation whenever the system becomes empty. A queue of arriving customers will form while the server is on vacation, but as soon as that queue grows to N customers, the server will immediately return and begin serving customers.
Let $B^*(s)$ be the service time transform and $G^*(s)$ be the busy period transform for an ordinary M/G/1 system (with no vacations) and let $\hat{G}^*_N(s)$ be the busy period transform for the lazy M/G/1 system (with vacations).
(a) (2%) Give the equation for $G^*(s)$.
(b) (1%) For what value of N does the lazy M/G/1 become an ordinary M/G/1?
(c) (4%) Find $\hat{G}^*_N(s)$ in terms of $B^*(s)$, $G^*(s)$, λ and N.
(d) (2%) From (a) and (c), express $\hat{G}^*_N(s)$ in terms of $G^*(s)$ and N only.
(e) (3%) Suppose now that N is a random variable with P_n = P[N=n] and transform $P(Z) = \sum_{n=1}^{\infty} P_n Z^n$. Express $\hat{G}^*_N(s)$ in terms of $P(Z)$ and $G^*(s)$.

3. (14%) We have a G/M/1 queueing system with $\mu = 1$ and $A^*(s) = \frac{a}{(s+3/2)(s+1)}$.
(a) (1%) Find the value for a.
(b) (2%) Find ρ.
(c) (4%) Find the critical system parameter σ.
(d) (2%) Find r_k = P[arrival finds k in system].
(e) (2%) Find $W(y) = P[\tilde{w} \leq y]$.
(f) (3%) Find P_0 = P[system is empty] and compare P_0 to d_0=P[departure leaves behind empty system].

4. (12%) Consider the following G/G/1 system:

$$A^*(s) = \frac{2}{(2s+1)(s+2)}$$
$$B^*(s) = \frac{1}{1+2s}$$

(a) (4%) Find $\frac{\Psi_+(s)}{\Psi_-(s)}$
. (b) (4%) Draw the pole-zero plot for $\frac{\Psi_+(s)}{\Psi_-(s)}$.
(c) (4%) Find $\Phi_+(s)$.
(d) (4%) Find $W(y) = P[\tilde{w} \leq y]$.

5. (6%) Suppose we have two different queueing systems indexed by i (i=1,2). We are told the following:

$$A_1^*(s) = B_2^*(s) = \frac{2}{s+2}$$
$$W_1^*(s) = W_2^*(s)$$

(a) (3%) How must $A_2^*(s)$ and $B_1^*(s)$ be related?
(b) (3%) Find the 49th moment of the inter-arrival time for the first system.

6. (12%) We have an M/M/1 queueing system with parameters λ and μ. However, at each of the arrival points we may have more than one arrival. Specifically, we are equally likely to have 1, 2, or 3 arrivals. All customers are served separately.
(a) (3%) Draw the state diagram with labels.
(b) (3%) Let P_k = P[k in system in equilibrium]. Write down the balance equations for P_k (k=0,1,2,...).
(c) (2%) What value is ρ?
(d) (4%) Let $P(Z) = \sum_{k=0}^{\infty} P_k Z^k$. Find $P(Z)$ in terms of ρ and Z.

7. (12%) Consider an M/E_2/1 queueing system.
(a) (1%) What is the expression for the mean wait W in an M/G/1 queue?
(b) (3%) Find W for M/E_2/1 explicitly.
(c) (1%) Compare the mean wait in M/E_2/1 with the mean wait in M/M/1.
(d) (3%) Find T for M/E_2/1 explicitly.
(e) (1%) Compare $T_{M/E_2/1}$ with $T_{M/M/1}$.
(f) (1%) Find $\overline{N}$, the mean number in system, for M/E_2/1.
(g) (2%) Find $W^*(s)$ for M/E_2/1.

8. (12%) In class, we analyzed the series/parallel distributed processing system which consisted of m parallel chains of processors, with the ith chain (i = 1,2,3,...,m) consisting of n_i processors in series. In the ith chain, each processor has a capacity of C_i operations per second. The traffic carried by the ith chain is λ_i jobs per second, each job requiring an average of $1/\mu$ operations. For a job travelling down the ith chain, each processor in the chain processes only $1/n_i\mu$ operations. The system is fed with Poisson traffic at a rate λ jobs per second, and this traffic is

probabilistically split so that the ith chain carries an amount λ_i jobs per second. Job lengths are exponentially distributed and we model each processor as behaving as an M/M/1 queueing system. The total processing capacity of the system is C.
(a) (2%) What is the mean time it takes a job to pass through one processor in the ith chain (including queueing time)?
(b) (1%) Let T_i be the mean time to pass down the ith chain. Find T_i.
(c) (2%) In terms of the defined quantities, give a definition for T, the mean response time of a job which arrives to this series/parallel distributed system.
(d) (1%) Find T in terms of the underlying system parameters.
(e) (2%) Now assume that n_i = n and $\lambda_i = \lambda$/m for all i = 1,2,...,m and that C_i = C/mn. Find T. Let us call this T(mn).
(f) (1%) Suppose m=n=1. Clearly this is a purely centralized system. For this system, find T, and define this to be T(1).
(g) (3%) For this part, we will hold C constant for the centralized system, but will consider that the configuration of part (e) has a capacity equal to A*C. Find A such that $T(mn) = T(1)$. Derive your answer.

9. (8%) Consider M independent M/M/1 queueing systems each with parameters λ and μ. Let $\rho=\lambda/\mu$. We wish to calculate F_M=E[fraction of wasted servers]
where a wasted server is an idle server who could be assigned to a waiting customer in one of the other queueing systems.
(a) (2%) Find F_1.
(b) (3%) Find F_2.
(c) (3%) Find F_3.