Midterm for Performance Analysis (closed book) April 20, 1998

Instructor: Prof. Ying-Dar Lin

1. (10%) Consider a two-state discrete time Markov chain. It has two states, 1 and 2, where the state transition probabilities from state 1 to state 1, from state 1 to state 2, from state 2 to 1, and from state 2 to state 2 are 3/4, 1/4, 1, and 0, respectively.
(1%)(a) Write down the probability transition matrix, P.
(3%)(b) Compute the matrix $(I - zP)^{-1}$.
(4%)(c) Find the time dependent state probabilities $\Pi(n) = [\pi_1(n), \pi_2(n)]$, in terms of $\pi_1(0)$, $\pi_2(0)$, and n.
(2%)(d) Find the equilibrium state probabilities $\Pi = [\pi_1, \pi_2]$.

2. (15%)Suppose Albert and Betsy run a race repeatedly. The times required for Albert and Betsy to complete the race, $\tilde{x}$ and $\tilde{y}$, are exponentially distributed with parameters α and β, respectively.
(3%)(a) Assume $\tilde{z} = min[\tilde{x}, \tilde{y}]$. Find the distribution of $\tilde{z}$, i.e. $P[\tilde{z} \leq z]$.
(3%)(b) Find $P[\tilde{x} < \tilde{y}]$.
(4%)(c) Let $\tilde{n_b}$ denote the number of times Betsy wins before Albert wins his first race. Find $P[\tilde{n_b} = n]$ for $n \geq 0$.
(5%)(d) Suppose $\tilde{w} = \sum_{i=1}^{\tilde{n_b}} \tilde{z}_i$ where $\tilde{z}_i$ is a sequence of z drawn from (a) and $\tilde{n_b}$ is defined in (c). Find the distribution of $\tilde{w}$, i.e. $P[\tilde{w} \leq w]$.

3. (15%) Messages arrive to a statistical multiplexing system according to a Poisson process having rate λ. Message lengths, denoted by $\tilde{m}$, are specified in octets, groups of eight bits, and are drawn from an exponential distribution having mean $1/\mu$. Messages are multiplexed onto a single trunk having a transmission capacity of C bits per second according to a FCFS discipline.
(3%)(a) Let $\tilde{x}$ denote the time required for transmission of a message over the trunk. Find the distribution of $\tilde{x}$, i.e. $P[\tilde{x} \leq x]$. (Note the difference between bits and bytes.)
(3%)(b) Suppose $E[\tilde{m}] = 128$ octets and $C = 56$ kilobits per second (Kbps). Determine λ_{max}, the allowed maximum message arrival rate.
(3%)(c) Let $\tilde{n}$ denote the number of messages in the system in stochastic equilibrium. Under the condition of (b), determine $P[\tilde{n} > n]$ as a function of λ.
(3%)(d) From (c), determine the maximum value of λ such that $P[\tilde{n} > 50] < 10^{-2}$.
(3%)(e) For the value of λ determined in (d), compute the mean value of $\tilde{y}$, the busy period of the system.

4. (11%) Now consider a regular M/M/1 system with service rate μ and arrival rate λ, with $\lambda < \mu$. Recall our notation where $\tilde{t}$ is the interarrival time with p.d.f. $a(t)$ and Laplace transform $A^*(s)$. For the service time, $\tilde{x}$, we have $b(x)$ and $B^*(s)$. Let $\tilde{d}$ be the interdeparture time with p.d.f. $d(t)$ and transform $D^*(s)$. For M/M/1, it is known that $1 - \rho =$ P[departure leaves behind an empty system].
(2%)(a) Express $\tilde{d}$ in terms of $\tilde{t}$, $\tilde{x}$, and ρ.

(2%)(b) Express $d(t)$ in terms of $a(t)$, $b(t)$, and ρ. (hint: use conditional prob and convolution)
(2%)(c) Express $D^*(s)$ in terms of $A^*(s)$, $B^*(s)$, and ρ.
(2%)(d) For this M/M/1 system, what are the expressions for $A^*(s)$ and $B^*(s)$?
(2%)(e) From (c) and (d), find $D^*(s)$ explicitly.
(1%)(f) From (e), find $d(t)$ explicitly.

5. (13%) We have an M/M/1 queueing system with parameters λ and μ. However, at each of the arrival points we may have more than one arrival. Specifically, we are equally likely to have 1, 2, or 3 arrivals. All customers are served separately.
(3%)(a) Draw the state transition diagram with labels.
(3%)(b) Let p_k = P[k in system in equilibrium]. Write down the balance equations for p_k (k = 0,1,2, ...), assuming $\rho < 1$.
(2%)(c) What value does ρ have?
(5%)(d) Let $P(z) = \sum_{k=0}^{\infty} p_k z^k$. Find $P(z)$ in terms of ρ and z.

6. (16%) Consider an M/M/1/1 system where the arrival process is Poisson with rate λ and the service rate is exponential with rate μ, and the maximum occupancy is 1. Let $P(t)$ be the row vector of time dependent state probabilities, $P(t) = [P_0(t), P_1(t)]$. Q is the infinitesimal generator matrix satisfying $\frac{d}{dt}P(t) = P(t)Q$.
(3%)(a) Write down Q.
(6%)(b) Find the eigenvalues of Q and their corresponding eigenvectors.
(4%)(c) Compute $P(t)$, in terms of $P(0)$ (i.e. $P_0(0)$, $P_1(0)$), λ, μ, and t.
(3%)(d) Find $\lim_{t\to\infty} P(t)$.

7. (20%) Consider a PH/PH/1 system where the system alternates between the ON and OFF phases, which have exponential duration with parameters α and β, respectively. During the ON periods, the system has Poisson arrival with rate λ. During the OFF periods, the system has no arrival. The service process of the system is always exponential with rate μ, regardless of the system phase.
(2%)(a) Define the state variable for this quasi-birth-deadth process.
(3%)(b) Draw the state transition diagram with labels.
(4%)(c) Write down the balance equations.
(3%)(d) Convert the equations in (c) into the matrix format.
(5%)(e) Define $G(z) = \sum_{n=0}^{\infty} z^n P_n$, i.e. $G(z) = [G_0(z), G_1(z)]$ and $G_i(z) = \sum_{n=0}^{\infty} z^n P_{n,i}$, $i = 0, 1$. If $\alpha = \beta = \lambda = \mu = 1$, find $G(z)$ and P_0.
(3%)(f) From (e), find $P_n e$, in terms of n.