Final for Probability (closed book) June 11, 2001

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Total points: 100

1. (12%) A binary communications channel has equiprobable inputs, 0 and 1. Input 1 has an error probability σ for output, i.e. input 1 becomes output 0 with probability σ. However, input 0 has zero error probability for output.
(4%)(a) Find the probability that the output is 0.
(4%)(b) Find the probability that the input was 0 given that the output is 0.
(4%)(c) Find the probability that the input was 1 given that the output is 0.

2. (15%) For an exponential random variable X with parameter λ, do the following.
(3%)(a) Write down X's pdf.
(4%)(b) Find the characteristic function of X, i.e. the Laplace transform of X's pdf.
(4%)(c) From the Laplace transform in (b), find X's mean.
(4%)(d) From the Laplace transform in (b), find X's variance.

3. (13%) A factory has n machines of a certain type. Let p be the probability that a machine is working on any given day, and let N be the total number of machines working on a certain day. The time, T, required to manufacture an item is an exponentially distributed random variable with parameter $i\alpha$ if i machines are working.
(4%)(a) Find P[N=i].
(4%)(b) Find P[$T \leq t \parallel N = i$].
(5%)(c) Find P[$T \geq t$].

4. (13%) The random variables X and Y have the joint pdf $f_{x,y}(x, y)$=$2e^{-(x+y)}$, $0 \leq x,\ y \leq \infty$.
(4%)(a) Find the marginal pdf $f_x(x)$ and $f_y(y)$.
(5%)(b) If Z=X+Y, find the cdf of Z, i.e. $F_z(z)$.
(4%)(c) Find the pdf of Z, i.e. $f_z(z)$.

5. (14%) A student uses pens whose lifetime is an exponential random variable with parameter 1, i.e. with mean 1 week. Suppose he buys n pens at the beginning of a semester, and S_n is the number of weeks his n pens could last.
(3%)(a) Find the mean number of weeks his pens could last.
(3%)(b) Find the variance of the number of weeks his pens could last.
(4%)(c) If a semester has 15 weeks, using the central limit theorem, find P[S_n ¿ 15], in terms of n.
(4%)(d) From a table lookup, $Q(-2.3263)$=0.99. Find the minimum n required so that with probability 0.99 he does not run out of pens during the semester.

6. (10%) Customers arrive at a soft drink dispensing machine according to a Poisson process with

rate λ. Let $N(t)$ be the number of customer arrivals up to time t. Suppose that each time a customer deposit money, the machine dispenses a random number of soft drinks. This random number is a Poisson random variable with parameter 1. Let $X(t)$ be the number of drinks dispensed up to time t.
(5%)(a) Find $P[X(t)=j \parallel N(t)=n]$.
(5%)(b) Find $P[X(t)=j]$.

7. (15%) A discrete-time Markov chain has two states, state 0 and state 1. At state 0, the probabilities to stay in state 0 and to jump to state 1 are both 1/2. At state 1, the probabilities to stay in state 1 and to jump to state 0 are 3/4 and 1/4, respectively.
(3%)(a) Sketch the state transition diagram of this Markov chain.
(3%)(b) Write down the one-step transition matrix.
(5%)(c) If the initial state of this Markov chain is at state 0, find the probabilities of being at state 0 and state 1, repectively, after 2 time units.
(4%)(d) Find the steady-state probabilities of being at state 0 and state 1, respectively.

8. (8%) Consider an M/M/1 queueing system, with arrival rate λ and service rate μ, in which each customer arrival brings in a profit of 5 dollors but in which each unit time of delay costs the system 1 dollor.
(4%)(a) Find the mean time units of delay for customers.
(4%)(b) Find the relationship between λ and μ for which the system makes a net profit.